Discrete-time trawl processes

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Abstract

We introduce a class of discrete time stationary trawl processes taking real or integer values and written as sums of past values of independent 'seed' processes on shrinking intervals ('trawl heights'). Related trawl processes in continuous time were studied in Barndorff-Nielsen et al. (2011, 2012).

In the case when the trawl function decays as a power function of the lag with exponent $1 < \alpha < 2$, the trawl process exhibits long memory and its covariance function is non-summable. We show that under general conditions on generic seed process, the normalized partial sums of such trawl process may tend either to a fractional Brownian motion or to an α-stable Lévy process. Moreover if the trawl function admits a faster decay rate, then the classical Donsker's invariance principle holds true.

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1. Introduction

The present paper introduces a class of stationary random processes of the form

$$X_k = \sum_{j=0}^{\infty} \gamma_{k-j}(a_j), \qquad k \in \mathbb{Z} \tag{1.1}$$

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where $\gamma_k = \{\gamma_k(u), u \in \mathbb{R}\}$, for $k \in \mathbb{Z}$, are i.i.d. copies of a generic process $\gamma = \{\gamma(u),\ u \in \mathbb{R}\}$ tending to zero in probability as $u \to 0$, and $a_j, j \ge 0$ is a sequence of real numbers satisfying $\lim_{j\to\infty} a_j = 0$. Throughout this paper, we use standard notation $\mathbb{N} = \{0, 1, \dots\}$, $\mathbb{Z} = \{0, \pm 1, \dots\}$, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty), u \wedge v = \min\{u, v\}$. Clearly, (1.1) includes the class of causal moving averages $X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}$ in i.i.d. r.v.s $\{\xi_k,\ k \in \mathbb{Z}\}$, which correspond to a random line seed process $\gamma = \{\gamma(u) = \xi_0 u,\ u \in \mathbb{R}\}$.

In the sequel we call $X = \{X_k,\ k \in \mathbb{Z}\}$ in (1.1) the *trawl process* corresponding to a *seed process* $\gamma = \{\gamma(u),\ u \in \mathbb{R}\}$ and a *trawl (function)* $a = \{a_j,\ j \ge 0\}$. The above terminology is borrowed from Barndorff-Nielsen et al. [3], where a related class of trawl processes in continuous time was introduced. To be more specific, [3] consider stochastic integrals

$$Y_t \;=\; \int_{\mathbb{R}\times(-\infty,t]} \mathbb{1}(x \in (0, d_{t-s}))L(\mathrm{d}x, \mathrm{d}s), \qquad t \in \mathbb{R} \tag{1.2}$$

where $L(\mathrm{d}x, \mathrm{d}s)$ is a homogeneous Lévy *basis* on $\mathbb{R}^2$ and $\{d_t, t \in \mathbb{R}_+\}$ is a deterministic function satisfying certain conditions. In the case when this function takes constant values on intervals $t \in (j, j+1]$, for $j = 0, 1, \dots$, the discretized process $\{Y_k,\ k \in \mathbb{Z}\}$ in (1.2) coincides with $\{X_k,\ k \in \mathbb{Z}\}$ in (1.1) corresponding to the independent increment (Lévy) seed process and to the trawl function

$$\Big\{\gamma(u) = \int_{(0,u]\times(0,1]} L(\mathrm{d}x, \mathrm{d}s),\ u \in \mathbb{R}\Big\}, \qquad \Big\{a_j = d_t,\ t \in (j, j+1],\ j \ge 0\Big\}.$$

Clearly, an integer-valued seed process $\gamma = \{\gamma(u),\ u \in \mathbb{R}\}$ in (1.1) results in an integer-valued trawl process $\{X_k,\ k \in \mathbb{Z}\}$, similarly as in the case of continuous-time trawl processes of (1.2) studied in [3]. On the other hand, the discrete-time set-up allows us to consider very general seed processes γ which need not be infinitely divisible or have independent increments as in the aforementioned work.

Barndorff-Nielsen et al. ([2], p. 22) note that trawl processes represent a flexible class of stochastic processes which can be used to model serially dependent count data and other stationary time series, where the marginal distribution and the autocorrelation structure can be modelled independently from each other. In particular, trawl processes can exhibit long memory or long-range dependence, which is usually associated with divergence of covariance series: $\sum_{k\in\mathbb{Z}} |\mathrm{Cov}(X_0, X_k)| = \infty$, see [11], and which occurs in models (1.1) and (1.2) when the trawl function decays sufficiently slowly with the lag, see [3] and Section 2 below. Fig. 6 in [3] shows sample paths and autocorrelation graphs of integer-valued trawl process given by (1.2) with $d_t = (1-t)^{-1.03}$, exhibiting a remarkably slow decay of the theoretical and the sample ACFs for lags up to 1000.

The main question studied in this paper, which is also one of the basic questions for statistical applications of trawl processes, is the rate of convergence and the limit distribution of the sample mean. We prove that for trawl process with trawl function a_j decaying as $j^{-\alpha}$ $(j \to \infty)$, for $1 < \alpha < 2$ this limit distribution may be either α-stable or Gaussian. Moreover, a non-Gaussian stable limit is typical for integer valued seed (and trawl) process, while a Gaussian limit occurs for 'continuous' seed processes, e.g. diffusions or stochastic volatility processes. See Theorems 1 and 2 below for precise statements. We note that our non-Gaussian result contradicts the conjecture in ([3], p. 708) about Gaussian partial sums limit for long-memory trawl process in (1.2). In particular, for a standard Poisson seed process γ and $0 \le a_j \sim c_0 j^{-\alpha}$, $1 < \alpha < 2$, $c_0 > 0$, we prove that the partial sums process $S_{[nt]} = \sum_{j=1}^{[nt]}(X_j - \mathbb{E}X_j)$, when normalized by $n^{1/\alpha}$, tends to an α-stable Lévy process weakly in the Skorohod space equipped with M_1-topology, see Theorem 3 below, and at the same time the covariance $\mathrm{Cov}(n^{-H}S_{[nt]}, n^{-H}S_{[ns]}) \sim (c/2)(t^{2H} + s^{2H} - |t-s|^{2H})$ approaches the covariance of fractional Brownian motion with variance ct^{2H}, $c > 0$ and Hurst index $H = (3-\alpha)/2 > 1/\alpha$. However if a_j decay as $O(j^{-\alpha}), \alpha > 2$ the Donsker functional central limit theorem holds for the partial sums process, with usual Brownian limit and $\sqrt{n}$-normalization.

A similar phenomenon (weak convergence of the partial sums process to a Lévy stable process) occurs for a number of long-range dependent stationary processes with finite variance, see [28], [33], [17], [26], [35], [21], [32], [16], [27] and the references therein. We note that in most of the literature this convergence is limited to finite-dimensional distributions. For $M/G/\infty$ queue with heavy-tailed activity periods, the adequate functional convergence was proved in [29]. Since the limiting stable processes in these works have independent increments, the above behavior is sometimes called 'distributional short-range dependence' in contrast to 'distributional long-range dependence' occurring when the limit of the partial sums process has dependent increments. See [7], [20]. See also [22] for a nice discussion of stable and Gaussian limits under long-range dependence.

The results of this paper concern linear functionals (partial sums) of trawl processes. For many statistical applications, limit theorems for nonlinear functionals (e.g., quadratic forms, empirical processes) are needed. For some classes of long memory processes (which include the linear process and the infinite source Poisson transmission model), these questions were addressed in [13], [8], [11], [30] and other works. A useful property of trawl processes corresponding to Poisson and some other jump-type seed processes is association, see Section 3.3, which might facilitate the study of limit theorems for certain nonlinear functionals. See [23] for weak convergence of empirical process under association.

2. Discrete-time trawl process

2.1. Existence of discrete-time trawl process

Let $\gamma_k = \{\gamma_k(u), u \in \mathbb{R}\}$, $k \in \mathbb{Z}$ be i.i.d. copies of a (generic) seed process $\gamma = \{\gamma(u), u \in \mathbb{R}\}$ with finite variance $\rho(u) = \mathrm{Var}(\gamma(u))$ and mean

$\mu(u) = \mathbb{E}\gamma(u)$. A trawl $a = \{a_j,\, j \ge 0\}$ is a deterministic sequence such that $\lim_{j\to\infty} a_j = 0$. We shall assume that

$$\mathbb{E}\gamma(u) = O(\mathrm{Var}(\gamma(u))) \to 0, \quad u \to 0, \tag{2.3}$$

and

$$\sum_{j=0}^{\infty} \mathrm{Var}(\gamma(a_j)) < \infty. \tag{2.4}$$

The trawl process $X = \{X_k,\, k \in \mathbb{Z}\}$ corresponding to trawl $a = \{a_j,\, j \ge 0\}$ and seed process $\gamma = \{\gamma(u), u \in \mathbb{R}\}$ is defined in (1.1).

Let

$$\rho(u,v) = \mathrm{Cov}(\gamma(u), \gamma(v)), \quad \rho(u) = \rho(u,u), \quad u, v \in \mathbb{R} \tag{2.5}$$

denote the covariance and the variance of the seed process.

Proposition 1. *Let conditions* (2.3) *and* (2.4) *be satisfied. Then the series in* (1.1) *converges a.s. and in mean square for any* $k \in \mathbb{Z}$. *Moreover* $\{X_k, k \in \mathbb{Z}\}$ *in* (1.1) *defines a stationary process with mean* $\mathbb{E}X_k = \sum_{j=0}^{\infty} \mu(a_j)$ *and covariance function*

$$\mathrm{Cov}(X_0, X_k) \;=\; \sum_{j=0}^{\infty} \rho(a_j, a_{j+k}), \qquad k \in \mathbb{N}. \tag{2.6}$$

Proof. The convergence of (1.1) is an easy consequence of the Kolmogorov three series theorem. Stationarity of (1.1) follows from the fact that the distribution of $\{\gamma_{k+h-j}(a_j), k \in \mathbb{Z}, j \in \mathbb{N}\}$ does not depend on $h \in \mathbb{Z}$. □

Clearly, if the seed process takes integer values: $\gamma(u) \in \mathbb{Z}, u \in \mathbb{R}$, this property also holds for the trawl process: $X_k \in \mathbb{Z}$ $(\forall\, k \in \mathbb{Z})$. The following examples show that the class of trawl processes is very large.

Example 1 (Random line seed process)**.** Let $\gamma(u) = \xi u, u \in \mathbb{R}$, where ξ is a r.v. with zero mean and variance $\sigma^2 < \infty$. Then $\mu(u) = 0$, $\rho(u) = \sigma^2 u^2$, condition (2.3) holds trivially and condition (2.4) translates to $\sum_{j=0}^{\infty} a_j^2 < \infty$. Then X in (1.1) is a moving-average:

$$X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}, \tag{2.7}$$

where $\{\xi_k, k \in \mathbb{Z}\}$ are i.i.d. copies of ξ.

Example 2 (Brownian motion seed process)**.** Let $a_j \ge 0$ and $\gamma(u) = B(u)$, $u \ge 0$, where B is a Brownian motion with zero mean and covariance $\mathbb{E}B(u)B(v) = u \wedge v$. Then (2.3) is trivially satisfied while (2.4) becomes $\sum_{j=0}^{\infty} a_j < \infty$. Then

X in (1.1) is a stationary Gaussian process with zero mean and covariance $\mathrm{Cov}(X_0, X_k) = \sum_{j=0}^{\infty} a_j \wedge a_{k+j}$, $k \in \mathbb{N}$. Particularly, if $a_j = a^j$, $a \in (0,1)$ then $\mathrm{Cov}(X_0, X_k) = a^k/(1-a)$ and finite-dimensional distributions X in (1.1) coincide with those of an AR(1) process written as a moving-average in (2.7) with $a_j = a^j$ and Gaussian i.i.d. innovations $\xi_k \sim \mathcal{N}(0, \sigma^2)$, where $\sigma^2 = 1 + a$.

Example 3 (Poisson and mixed Poisson seed processes)**.** Let $\gamma(u) = P(u)$, $u \in \mathbb{R}_+$, where P is a Poisson process with mean $\mu(u) = u$, covariance $\rho(u,v) = \mathrm{Cov}(P(u), P(v)) = u \wedge v$ and $a_j \ge 0$, $\sum_{j=0}^{\infty} a_j < \infty$. Then (2.3) and (2.4) are satisfied since $\mu(u) = \rho(u)$ and X in (1.1) is a stationary process with mean $\mathbb{E}X_k = \sum_{j=0}^{\infty} a_j$ and the same covariance as in Example 2. Moreover, X_k takes integer values and has a Poisson marginal distribution with mean $\mathbb{E}X_0$.
The above example can be generalized by considering a mixed Poisson seed process $\gamma(u) = P(u\zeta)$, where P is as above and $\zeta \ge 0$ is a random variable with $\mathbb{E}\zeta < \infty$, independent of P. Particularly, [5] proved that when ζ is exponentially distributed then $P(u\zeta)$ has negative binomial marginal distribution. The case of binary r.v. $\zeta \in \{0,1\}$ corresponds to the so-called zero-inflated Poisson process, see [19]. Note that for $\gamma(u) = P(u\zeta)$

$$\mu(u) = u\mathbb{E}\zeta \quad \text{and} \quad \rho(u,v) = (u \wedge v)\mathbb{E}\zeta + uv\mathrm{Var}(\zeta).$$

Example 4 (Bernoulli and binomial seed processes)**.** The Bernoulli seed process is defined by $b(u) = \mathbb{1}(U \le u)$, where $U \sim \mathcal{U}[0,1]$ is a uniformly distributed random variable. Thus, for $\gamma(u) = b(u)$

$$\mu(u) = u, \quad \rho(u,v) = u \wedge v - uv.$$

The binomial seed $\gamma(u) = b(u;n), u \ge 0$ is defined as the sum of n independent Bernoulli seeds: $b(u;n) = \sum_{j=1}^{n} b_j(u)$, where $b_j(u) = \mathbb{1}(U_j \le u), j = 1, \dots, n$ are independent Bernoulli processes. Clearly, $\mathbb{E}b(u;n) = nu$ and $\rho(u,v) = \mathrm{Cov}(b(u;n), b(v;n)) = n(u \wedge v - uv)$.

Further examples of trawl processes can be found in Sections 2.2 (Example 5), 3.1 (Examples 6-7) and 3.2 (Example 8).

2.2. Second order properties of discrete-time trawl process

The variance $\mathrm{Var}(X_k)$ of trawl process X in (1.1) depends both on trawl $a = \{a_j\}$ and on covariance function $\rho(u,v)$ of seed process, see (2.6). In order to characterize the existence of X in terms of $a = \{a_j\}$ alone, it is convenient to impose a linear growth condition on the variance $\rho(u) = \mathrm{Var}(\gamma(u))$ at the origin $u = 0$:

$$\rho(u) = O(|u|), \qquad u \to 0. \tag{2.8}$$

Under (2.8), condition (2.4) is equivalent to summability of the trawl sequence:

$$\sum_{j=0}^{\infty} |a_j| < \infty. \tag{2.9}$$

Clearly, the trawl processes in Examples 2-4 satisfy (2.8) provided the seed processes in these examples are suitably extended to negative $u < 0$.
The following proposition shows that trawl processes with seed process in these examples exhibit a rich covariance structure.

Proposition 2. *Let $r(k) \ge 0, k \in \mathbb{N}, \lim_{k\to\infty} r(k) = 0$ be a convex monotone function, viz., $r(k)-r(k+1) \ge 0$, $r(k+2)-2r(k+1)+r(k) \ge 0$, for $k \in \mathbb{N}$. Then $r(k) = \mathrm{Cov}(X_0, X_k), k \in \mathbb{N}$, where $\{X_k\}$ is the stationary trawl process in* (1.1) *with trawl function $a_j = r(j)-r(j+1) \ge 0$ and a seed process $\gamma = \{\gamma(u),\, u \ge 0\}$ such that $\mathbb{E}\gamma(u) = O(u)$, $\rho(u,v) = \mathrm{Cov}(\gamma(u),\gamma(v)) = u \wedge v$, $u, v \ge 0$.*

Proof. Since $\rho(u) = u$, $a_j \ge 0$ and $\sum_{j=0}^{\infty} a_j = r(0) < \infty$, so conditions (2.8) and (2.9) guaranteeing the existence of the corresponding trawl process (1.1) are satisfied. Then by (2.6) and monotonicity of a_j we have that $\mathrm{Cov}(X_0, X_k) = \sum_{j=k}^{\infty} a_j = r(k)$. □

Example 5. Covariance function of ARFIMA$(0,d,0)$ process with parameter $0 < d < 1/2$ is given by

$$\begin{aligned} r(j) &= r(0)\prod_{k=1}^{j} \frac{k-1+d}{k-d} = \frac{\Gamma(j+d)\Gamma(1-2d)}{\Gamma(j-d+1)\Gamma(d)\Gamma(1-d)} \\ &\sim \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} j^{-1+2d}, \quad j \to \infty, \end{aligned} \tag{2.10}$$

$r(0) = \Gamma(1-2d)/\Gamma^2(1-d)$, see e.g. ([11], (7.2.9)). Then (2.10) satisfies the conditions of Proposition 2: $r(j) - r(j+1) = r(j)\big(1 - \frac{j+d}{j+1-d}\big) = (1-2d)r(j)/(j+1-d) > 0$ and $r(j)-2r(j+1)+r(j+2) = 2(1-d)(1-2d)r(j)/(j+1-d)(j+2-d) > 0$, for $j \in \mathbb{N}$. Particularly, trawl process with Poisson seed process $\gamma(u) = P(u)$ in Example 3 and trawl $a_j = r(j) - r(j+1)$ defined by (2.10) presents an example of integer-valued process with Poisson marginal distribution and ARFIMA$(0,d,0)$ covariance function. Note that the above trawl decays as $j^{-2(1-d)}$ with the exponent $2(1-d) \in (1,2)$, viz.,

$$a_j = r(j)\frac{1-2d}{j+1-d} \sim \frac{\Gamma(2-2d)}{\Gamma(d)\Gamma(1-d)} j^{-2+2d}, \quad j \to \infty.$$

Denote by $S_n = \sum_{k=1}^{n} X_k$ the partial sums process of the trawl process in (1.1).
The following proposition obtains power-law decay of the covariance of the trawl

process and the asymptotics of the variance of S_n under general conditions on the trawl function and on the seed process. Contrary to Proposition 2 and Example 5, these conditions do not require monotonicity of a_j. Write $u_n \gg v_n$ for $\lim_{n\to\infty} u_n/v_n = \infty$.

Proposition 3. *Consider the stationary trawl process* $\{X_k\}$ *in* (1.1)*. Let conditions* (2.3)*,* (2.8) *and* (2.9) *be satisfied.*

(i) In addition, assume

$$\rho(u,v) \;=\; (|u| \wedge |v|)(1+o(1)), \quad \textit{as} \quad u,v \to 0,\; uv > 0. \tag{2.11}$$

and

$$a_j \;=\; c_0 j^{-\alpha}(1+o(1)), \quad j \to \infty \quad (\exists\, 1 < \alpha < 2,\; c_0 \neq 0). \tag{2.12}$$

Then

$$\mathrm{Cov}(X_0, X_k) \;=\; c_1 k^{1-\alpha}(1+o(1)), \qquad k \to \infty \tag{2.13}$$

and

$$\mathrm{Var}(S_n) = \sum_{k,l=1}^{n} \mathrm{Cov}(X_k, X_l) \;\sim\; c_2\, n^{3-\alpha} \gg n, \quad n \to \infty, \tag{2.14}$$

where $c_1 = |c_0|/(\alpha-1), c_2 = 2c_1/(2-\alpha)(3-\alpha)$.

(ii) In addition, assume

$$|\rho(u,v)| \le C(|u| \wedge |v|), \qquad u, v \in \mathbb{R}, \tag{2.15}$$

and

$$\sum_{j=1}^{\infty} j|a_j| \;<\; \infty. \tag{2.16}$$

Then

$$\sum_{k=1}^{\infty} |\mathrm{Cov}(X_0, X_k)| \;<\; \infty \tag{2.17}$$

and

$$\mathrm{Var}(S_n) = n \sum_{|k|<n} \Big(1 - \Big|\frac{k}{n}\Big|\Big) \mathrm{Cov}(X_k, X_0) \sim \sigma^2\, n, \tag{2.18}$$

where $\sigma^2 = \sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_0, X_k)$.

Remark 1. (i) Note (2.11) and (2.12) imply (2.8) and (2.9), respectively. As noted above, the latter conditions together with (2.3) guarantee (2.4) and the existence of stationary trawl process (1.1) in Proposition 1.

(ii) In view of (2.13) and (2.10), the parameter $d = 1-\alpha/2 \in (0, 1/2)$ in Proposition 3 (i) can be identified as the long memory parameter of the trawl process X. Statistical estimation of this parameter presents considerable interest. We plan to study this question in a future work.

Proof. (i) Without loss of generality, let $c_0 > 0$ in (2.12); the proof in the case $c_0 < 0$ is analogous. Then $a_j > 0$, and $a_{k+j} > 0$ hold for all $k \geq 1$ and $j > j_0$, where j_0 is large enough. Moreover, for any $\epsilon > 0$ there exists $j_0 < j_\epsilon < \infty$ such that

$$a_{j+k} < a_j, \quad \text{for all} \quad j_\epsilon < j < k/2\epsilon, \text{ and } \ k \geq 2\epsilon j_\epsilon. \tag{2.19}$$

Indeed, by (2.12) we have that for any $\epsilon > 0$ there exists $j_\epsilon > j_0 > 0$ such that $a_j > c_0 j^{-\alpha}(1-\epsilon)$, $a_{k+j} < c_0(j+k)^{-\alpha}(1+\epsilon)$ and therefore

$$\Big(\frac{a_{j+k}}{a_j}\Big)^{\frac{1}{\alpha}} < \frac{j}{j+k}\Big(\frac{1+\epsilon}{1-\epsilon}\Big)^{\frac{1}{\alpha}}, \qquad \forall\, j > j_\epsilon, \quad \forall\, k \geq 1.$$

Since $((1+\epsilon)/(1-\epsilon))^{\frac{1}{\alpha}} < 1 + 2\epsilon$ if $\epsilon > 0$ is small enough, relation (2.19) follows since $j/(j+k) \leq 1/(1+2\epsilon)$ for $1 \leq j < k/2\epsilon$.
Consider (2.13). For sufficiently large k ($k > 2\epsilon j_\epsilon$) split $k^{\alpha-1}\mathrm{Cov}(X_0, X_k) = \sum_{j=0}^\infty k^{\alpha-1}\ \rho(a_j, a_{k+j}) = \sum_{i=1}^3 I_{i,k}$, where

$$I_{1,k} = \sum_{0 \leq j \leq j_\epsilon} \ldots, \qquad I_{2,k} = \sum_{j_\epsilon < j < k/2\epsilon} \ldots, \qquad I_{3,k} = \sum_{j \geq k/2\epsilon} \ldots.$$

By (2.12) and Cauchy-Schwartz inequality, for any fixed $\epsilon > 0$ and $1 \leq j \leq j_\epsilon$,

$$|\rho(a_j, a_{k+j})| \leq \rho(a_j)^{\frac{1}{2}} \rho(a_{k+j})^{\frac{1}{2}} \leq C|a_{k+j}|^{\frac{1}{2}} \leq Ck^{-\frac{\alpha}{2}}, \qquad k \to \infty$$

implying

$$|I_{1,k}| \leq Ck^{\alpha-1}k^{-\frac{\alpha}{2}} = O(k^{-(1-\frac{\alpha}{2})}) = o(1), \qquad k \to \infty.$$

Next, by (2.11) and (2.12), $|\rho(a_j, a_{j+k})| \leq C|a_j| \wedge |a_{j+k}| \leq Cj^{-\alpha}$, ($\forall\, j, k \geq 1$) and therefore

$$I_{3,k} \leq Ck^{\alpha-1} \sum_{j \geq k/2\epsilon} j^{-\alpha} \leq C\epsilon^{\alpha-1}$$

can be made arbitrarily small uniformly in $k \geq 1$ by choosing $\epsilon > 0$ small enough. Finally, by (2.19) and (2.11),

$$I_{2,k} = c_0 k^{\alpha-1} \sum_{j_\epsilon < j < k/2\epsilon} \frac{1 + \delta_{j,k}}{(k+j)^\alpha}, \tag{2.20}$$

where $\sup_{j \geq 1} |\delta_{j,k}| = 0$ as $k \to \infty$. Note that for each $\epsilon > 0$, as $k \to \infty$

$$\begin{aligned} J_k(\epsilon) \ &:= \ k^{\alpha-1} \sum_{j_\epsilon < j < k/2\epsilon} (k+j)^{-\alpha} \ = \ \frac{1}{k} \sum_{\frac{j_\epsilon}{k} < \frac{j}{k} < 1/2\epsilon} \frac{1}{\big(1 + \frac{j}{k}\big)^\alpha} \\ &\to \ \int_0^{1/2\epsilon} \frac{\mathrm{d}x}{(1+x)^\alpha} \ = \ \frac{1}{\alpha-1}\big(1 - (2\epsilon)^{\alpha-1}\big). \end{aligned} \tag{2.21}$$

According to (2.20) and (2.21), for any $\delta > 0$ and any $\epsilon_0 > 0$ one can find $0 < \epsilon < \epsilon_0$ and $K_0 > 0$ such that $|I_{2,k} - c_0/(\alpha-1)| < \delta$ holds for all $k > K_0$.

This proves (2.13) while (2.14) follows from (2.13), see e.g. ([11], Proposition 3.3.1).

(ii) It suffices to prove (2.17) since (2.18) follows from (2.17) and the dominated convergence theorem. According to (2.6), (2.15), (2.16),

$$\begin{aligned}\sum_{k=1}^{\infty}|\mathrm{Cov}(X_0,X_k)| &\le C\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}|a_j|\wedge|a_{j+k}| \\ &\le C\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}|a_{j+k}| \le C\sum_{k=1}^{\infty}k|a_k| < \infty.\end{aligned}$$

Proposition 3 is proved. □

3. Partial sums limits of trawl processes

This section discusses partial sums limits for trawl processes in (1.1) satisfying the conditions of Proposition 3. Particularly, we detail conditions on the seed process $\{\gamma(u), u \in \mathbb{R}\}$ which guarantee that the partial sums process of the trawl process $\{X_k\}$ with regularly decaying trawl (2.12) tends to either a Gaussian process (fractional Brownian motion with Hurst parameter $H = (3-\alpha)/2 \in (1/2, 1)$) or to an α-stable Lévy process.

The following decomposition of the partial sums process as a sum of independent random variables is crucial for the proofs of Theorem 1 and Theorem 2.

Lemma 1 (Decomposition). *Let* $\{X_k\}$ *be as in* (1.1). *Then* $S_n = \sum_{k=1}^{n} X_k = \sum_{s=-\infty}^{n} Z_{s,n}$, *where*

$$Z_{s,n} = \sum_{k=1\vee s}^{n}\gamma_s(a_{k-s}), \qquad -\infty < s \le n \tag{3.22}$$

are independent r.v.s.

The proof of Lemma 1 follows trivially from the definition of X_k and the independence of the sequence $(\gamma_s)_{s\in\mathbb{Z}}$. Write $\to_{f.d.d.}$ for the weak convergence of finite-dimensional distributions and $\to_{\mathcal{D}(J_1)}$ and $\to_{\mathcal{D}(M_1)}$ for the weak convergence of random elements in the Skorohod space $D[0,1]$ endowed with the J_1- and M_1-topologies, respectively. For the definition of these topologies, see [31], [4], [24].

Denote $|\mu|_{2+\delta}(u) = \mathbb{E}|\gamma(u)|^{2+\delta}$ the absolute $(2+\delta)$-moment of the seed process.

3.1. Gaussian limit of the partial sums process

Theorem 1. *Consider a trawl process* $\{X_k\}$ *defined in* (1.1).

(i) Assume $\mu(u) = \mathbb{E}\gamma(u) = 0$, *(2.11), (2.12) and there exists* $\delta > 0$ *with*

$$|\mu|_{2+\delta}(u) = O(|u|^{\frac{2+\delta}{2}}), \qquad u \to 0. \tag{3.23}$$

Then

$$\frac{1}{n^H} S_{[nt]} \ \to_{\mathcal{D}(J_1)} \ \sqrt{c_2}\, B_H(t), \qquad H = \frac{3-\alpha}{2}, \tag{3.24}$$

where B_H *is a fractional Brownian motion with variance* $\mathbb{E}B_H^2(t) = t^{2H}$ *and* c_2 *is defined in* (2.14).

(ii) Assume $\mu(u) = \mathbb{E}\gamma(u) = 0$, *(2.15), (2.16), and (3.23).*
Then if also $\sigma^2 = \sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_0, X_k) \neq 0$, *we obtain:*

$$\frac{1}{\sqrt{n}} S_{[nt]} \ \to_{f.d.d.} \ \sigma\, B(t), \tag{3.25}$$

where B *is a Brownian motion with variance* $\mathbb{E}B^2(t) = t$.
In addition, if $\sum_{k=1}^{\infty} \sqrt{|a_k|} < \infty$, *then the finite dimensional convergence in* (3.25) *can be replaced by* $\to_{\mathcal{D}(J_1)}$.

(iii) All statements in (ii) remain valid if (3.23) *is replaced by*

$$|\mu|_{2+\delta}(u) = O(u) \quad (u \to 0), \quad \textit{and} \quad \sum_{j=0}^{\infty} |a_j|^{\frac{1}{2+\delta}} < \infty \quad (\exists\, \delta > 0). \tag{3.26}$$

Proof. We use the decomposition Lemma 1 and its notations. This is essential to use the Lindeberg theorem.
(i) Consider the convergence of one-dimensional distributions:

$$\frac{1}{\sqrt{n^{3-\alpha}}} S_n \ \to_{law} \ \mathcal{N}(0, c_2). \tag{3.27}$$

In view of (2.14) and Lemma 1, relation (3.27) follows by Lindeberg's theorem provided

$$L_n \ := \sum_{s=-\infty}^{n} \mathbb{E}|Z_{s,n}|^{2+\delta} \ = \ o\big(n^{\frac{(3-\alpha)(2+\delta)}{2}}\big). \tag{3.28}$$

The Lyapunov condition (3.28) seems to have been introduced quite early in the literature, see [25] or more recently ([12], theorem 3.5). By Minkowski's inequality and assumptions (2.9) and (3.23) we obtain

$$\begin{aligned} \mathbb{E}|Z_{s,n}|^{2+\delta} \ &\le \ \left(\sum_{k=1\vee s}^{n} (\mathbb{E}|\gamma(a_{k-s})|^{2+\delta})^{\frac{1}{2+\delta}} \right)^{2+\delta} \\ &\le \ C \left(\sum_{k=1\vee s}^{n} |a_{k-s}|^{\frac{1}{2}} \right)^{2+\delta} \ \le \ C \left(\sum_{k=1\vee s}^{n} |k-s|_+^{-\frac{\alpha}{2}} \right)^{2+\delta} \end{aligned} \tag{3.29}$$

(with $|\ell|_+ = \ell \vee 0$) and therefore $L_n \le C(L_n^- + L_n^+)$, where

$$L_n^- \;=\; \sum_{s=-\infty}^{0} \Big(\sum_{k=1}^{n} |k-s|_+^{-\frac{\alpha}{2}} \Big)^{2+\delta} \;=\; \sum_{s=0}^{\infty} \Big(\sum_{k=1}^{n} (k+s)^{-\frac{\alpha}{2}} \Big)^{2+\delta},$$

$$L_n^+ \;=\; \sum_{s=1}^{n} \Big(\sum_{k=1}^{n} k^{-\frac{\alpha}{2}} \Big)^{2+\delta} \;=\; n \Big(\sum_{k=1}^{n} k^{-\frac{\alpha}{2}} \Big)^{2+\delta}.$$

Here, $L_n^+ = O\big(n (n^{1-\frac{\alpha}{2}})^{2+\delta}\big) = o\big(n^{\frac{(3-\alpha)(2+\delta)}{2}}\big)$. The same relation for L_n^- follows from

$$L_n^- \;\le\; \int_0^\infty \mathrm{d}x \Big(\int_0^n (x+y)^{-\frac{\alpha}{2}} \mathrm{d}y \Big)^{2+\delta} \;=\; cn \big(n^{1-\frac{\alpha}{2}}\big)^{2+\delta}, \quad \text{with}$$

$$c \;=\; \int_0^\infty \mathrm{d}x \Big(\int_0^1 (x+y)^{-\frac{\alpha}{2}} \mathrm{d}y \Big)^{2+\delta} < \infty.$$

This proves (3.28) and the one-dimensional convergence in (3.27). Finite-dimensional convergence in (3.24) follows similarly using Cramér-Wold device. Finally, the tightness in $\mathcal{D}(J_1)$ of the partial sums process in (3.24) follows by Kolmogorov's criterion and from property (2.14) (see, e.g. [11], Proposition 4.2.2). This proves part (i).

(ii) Again, it suffices to prove the convergence of one-dimensional distributions:

$$n^{-1/2} S_n \;\to_{law}\; \mathcal{N}(0, \sigma^2). \tag{3.30}$$

By writing S_n as in (3.22) and using Lindeberg's theorem relation (3.30) follows from

$$L_n \;=\; \sum_{s=-\infty}^{n} \mathbb{E}|Z_{s,n}|^{2+\delta} \;=\; o\big(n^{\frac{2+\delta}{2}}\big). \tag{3.31}$$

Using Minkowski's inequality and assumptions (3.23) and (2.16) similarly as in part (i) we obtain

$$\mathbb{E}|Z_{s,n}|^{2+\delta} \;\le\; C\Big(\sum_{k=1\vee s}^{n} |a_{k-s}|^{\frac{1}{2}} \Big)^{2+\delta} \tag{3.32}$$

$$\le\; C\Big(\sum_{k=1\vee s}^{n} |(k-s)a_{k-s}| \Big)^{\frac{2+\delta}{2}} \Big(\sum_{k=1\vee s}^{n} (k-s)^{-1} \Big)^{\frac{2+\delta}{2}}$$

$$\le\; C\Big(\sum_{k=1\vee s}^{n} (k-s)^{-1} \Big)^{\frac{2+\delta}{2}}. \tag{3.33}$$

and hence

$$\sum_{s=-n}^{n} \mathbb{E}|Z_{s,n}|^{2+\delta} \;\le\; Cn(\log n)^{\frac{2+\delta}{2}} = o\big(n^{\frac{2+\delta}{2}}\big),$$

$$\sum_{s=-\infty}^{-n} \mathbb{E}|Z_{s,n}|^{2+\delta} \;\le\; C\sum_{s=n}^{\infty} \Big(\sum_{k=1}^{n} \frac{1}{k+s} \Big)^{\frac{2+\delta}{2}} \le C\sum_{s=n}^{\infty} (ns^{-1})^{\frac{2+\delta}{2}} \le Cn = o\big(n^{\frac{2+\delta}{2}}\big),$$

proving (3.31) and (3.30). To show the last statement of (ii), the tightness in $D[0,1]$, it suffices to prove the bound

$$\mathbb{E}|S_n|^{2+\delta} \ \le \ Cn^{\frac{2+\delta}{2}}, \tag{3.34}$$

see ([11], Proposition 4.4.4). By Rosenthal's inequality,

$$\mathbb{E}|S_n|^{2+\delta} \ \le \ C\Big(\sum_{s=-\infty}^{n} (\mathbb{E}|Z_{s,n}|^{2+\delta})^{\frac{2}{2+\delta}}\Big)^{\frac{2+\delta}{2}}.$$

Using (3.32) and $\sum_{k=1}^\infty |a_k|^{\frac{1}{2}} < \infty$, we get $\max_{|s|\le n} \mathbb{E}|Z_{s,n}|^{2+\delta} < C$ and

$$\begin{aligned}\sum_{s=-\infty}^{-n} (\mathbb{E}|Z_{s,n}|^{2+\delta})^{\frac{2}{2+\delta}} \ &\le \ C\sum_{s=n}^{\infty}\Big(\sum_{k=1}^{n}|a_{k+s}|^{\frac{1}{2}}\Big)^2 \\ &\le \ C\sum_{k_1,k_2=1}^{n}\sum_{s=n}^{\infty}|a_{k_1+s}|^{\frac{1}{2}}|a_{k_2+s}|^{\frac{1}{2}} \ \le \ Cn. \end{aligned} \tag{3.35}$$

This proves (3.34) and part (ii), too.

(iii) Similarly as in (3.29) and using (3.26) we get

$$\mathbb{E}|Z_{s,n}|^{2+\delta} \le C\Big(\sum_{k=1\vee s}^{n}|a_{k-s}|^{\frac{1}{2+\delta}}\Big)^{2+\delta} \le C\sum_{k=1\vee s}^{n}|a_{k-s}|^{\frac{1}{2+\delta}} \le C$$

for any $-\infty < s \le n$ and hence

$$\begin{aligned}\sum_{s=-\infty}^{-n}\mathbb{E}|Z_{s,n}|^{2+\delta} \quad &\le \ C\sum_{s=n}^{\infty}\sum_{k=1}^{n}|a_{k+s}|^{\frac{1}{2+\delta}} \ \le \ Cn, \\ \sum_{s=-\infty}^{-n}(\mathbb{E}|Z_{s,n}|^{2+\delta})^{\frac{2}{2+\delta}} \ &\le \ C\sum_{s=n}^{\infty}\big(\sum_{k=1}^{n}|a_{k+s}|^{\frac{1}{2+\delta}}\big)^2 \ \le \ Cn,\end{aligned}$$

as in (3.35). Hence, (3.31) and (3.34) follow, proving part (iii) and completing the proof of Theorem 1. □

Remark 2. The crucial condition for Gaussian partial sums limit under long-range dependence assumption (2.12) in Theorem 1 (i) is (3.23). Clearly this condition is satisfied for Brownian motion $\gamma(u) = B(u)$, in which case $|\mu|_{2+\delta}(u) = \mathbb{E}|B(u)|^{2+\delta} = |u|^{\frac{2+\delta}{2}}\mathbb{E}|B(1)|^{2+\delta}$. On the other hand, condition (3.23) is not satisfied for most jump processes. Particularly, if $\gamma(u) = P(u) - u, u \ge 0$ is a centered Poisson process with intensity $\mathbb{E}P(u) = u$, then

$$|\mu|_{2+\delta}(u) = u\mathrm{e}^{-u}|1-u|^{2+\delta} + O(u^{2+\delta}+u^2) \sim u, \quad u \to 0,$$

and (3.23) fails, but the first condition in (3.26) is satisfied. In particular, in the case of Poisson seed process, the trawl process satisfies Donsker's theorem if the trawl tends fast enough to 0 so that (3.26) holds.

Let us present further examples of seed processes satisfying the conditions in Theorem 1.

Example 6 (Geometric centered Brownian motion)**.** Set $\gamma(u) = \mathrm{e}^{B(u)-u/2} - 1, u \ge 0$, where B is a standard Brownian motion as above. We have $\mathbb{E}\gamma(u) = 0$ and, if $u \le v$,

$$\begin{aligned}\rho(u,v) &= \mathbb{E}\exp\{B(u)+B(v)-\frac{u+v}{2}\}-1\\ &= \exp\Big\{\Big(\frac{1}{2}\mathbb{E}\big(B(u)+B(v)\big)\Big)^2-\frac{u+v}{2}\Big\}-1\\ &= \exp\Big\{(\frac{1}{2}(u+v+2u)-\frac{u+v}{2}\Big\}-1\\ &= \mathrm{e}^u-1\\ &= u\wedge v+O\big((u\wedge v)^2\big), \quad u\wedge v\to 0.\end{aligned}$$

Therefore (2.11) is satisfied. We also have by Taylor's expansion that $|\mu|_4(u) = \mathbb{E}\left|\mathrm{e}^{B(u)-u/2}-1\right|^4 = \mathrm{e}^{6u}-4\mathrm{e}^{3u}+6\mathrm{e}^u-3 = O(u^2),\ u\to 0$ so that (3.23) is satisfied with $\delta = 2$.

Example 7 (Diffusion process)**.** Let

$$\gamma(u) = \int_0^u b(v)\mathrm{d}B(v), \qquad u\in\mathbb{R}_+,$$

where B is a Brownian motion, and $(b(v))_{v\in\mathbb{R}_+}$ is a random predictable process with $\lim_{v\to 0}\mathbb{E}b^2(v) = C > 0$. Then $\rho(u) = \int_0^u \mathbb{E}b^2(v)\mathrm{d}v \sim Cu\,(u\to 0)$ and $\rho(u,v) = \rho(u),\ 0\le u\le v$ so that (2.11) is satisfied. Moreover, if $\mathbb{E}|b(v)|^{2+\delta}\le C$ then by the moment inequality for Brownian integrals (see, e.g. [18], Theorem 9.9.2)

$$\begin{aligned}|\mu|_{2+\delta}(u) &\le C\mathbb{E}\Big(\int_0^u b^2(v)\mathrm{d}v\Big)^{\frac{2+\delta}{2}}\\ &\le C\Big(\int_0^u \mathbb{E}|b(v)|^{2+\delta}\mathrm{d}v\Big)\Big(\int_0^u 1\,\mathrm{d}v\Big)^{\frac{2+\delta}{2}-1} \;\le\; Cu^{\frac{2+\delta}{2}},\end{aligned}$$

hence assumption (3.23) holds, too.

3.2. Stable limit of the partial sums process

This subsection studies integer-valued trawl processes with seeds given by a general point process. We first discuss conditions on this point process guaranteeing the existence and stationarity of the trawl process. We assume that seed process $\gamma = \big\{\gamma(u),\, u\ge 0\big\}$ is a piecewise constant nondecreasing process

$$\gamma(u) = \sum_{k=0}^{\infty} k\cdot \mathbb{1}(\tau_k \le u < \tau_{k+1}) \tag{3.36}$$

starting at $\gamma(0) = 0$ with unit jumps at random points $0 = \tau_0 < \tau_1 \le \tau_2 \le \cdots \le \infty$. In particular, if $\tau_k < \tau_{k+1} = \cdots = \infty$, the number of jumps of γ does not exceed k and the process is bounded by k on $(0, \infty)$. We shall assume that the distribution of the first jump-point $\tau_1 > 0$ has a bounded probability density $\theta(\cdot)$:

$$\mathbb{P}(0 < \tau_1 \le u) = \int_0^u \theta(y)\mathrm{d}y, \qquad \text{and} \qquad \lim_{u\to 0} \theta(u) = 1. \tag{3.37}$$

Moreover, we shall suppose that there exists $\delta > 2(\alpha - 1)$ such that

$$\mathbb{E}\gamma(u)^{2+\delta} \ < \ \infty, \qquad\qquad \forall \ u > 0, \tag{3.38}$$

$$\mathbb{E}\gamma(u)^2 \mathbb{1}(\tau_2 \le u) \ = \ O(u^2), \qquad u \to 0. \tag{3.39}$$

Remark 3. The second condition in (3.37) can be replaced by $\lim_{u\to 0} \theta(u) = C > 0$ without loss of generality. Conditions (3.37)-(3.39) are very general and are satisfied by many jump processes, see Example 8 below. Note that conditions (3.37)-(3.39) as well as (3.40) given below, refer to the first two jump-times $0 < \tau_1 < \tau_2$ and do not involve subsequent jumps τ_k, for $k \ge 3$. As shown below, these conditions imply the existence and stationarity of the trawl process for general trawls.

Observe that $(\tau_1 \le u) = (\gamma(u) \ge 1)$, $(\tau_2 \le u) = (\gamma(u) \ge 2)$ and therefore an alternative way to set condition (3.39) is $\mathbb{E}\gamma^2(u)\mathbb{1}(\gamma(u) > 1) = O(u^2)$, as $u \to 0$.

Proposition 4. *(i) For the seed process γ in* (3.36)*, conditions* (3.37)-(3.39) *imply the assumptions* (2.3) *and* (2.4)*. In particular, the corresponding trawl process in* (1.1) *is stationary, has finite variance and the covariance in* (2.6)*, for any trawl $\{a_j \ge 0\}$ satisfying the summability condition in* (2.9)*.*

(ii) In addition to (3.37)-(3.39)*, assume that*

$$\mathbb{E}\gamma(v)\mathbb{1}(\tau_1 \le u, \tau_2 \le v) \ = \ o(u), \qquad 0 \le u \le v \to 0. \tag{3.40}$$

Then (2.11) *is satisfied. As a consequence, for regularly decaying trawl as in* (2.12) *the corresponding stationary trawl process $\{X_k\}$ in* (1.1) *enjoys the long memory properties in* (2.13) *and* (2.14)*.*

Proof. (i) We shall prove that $\mu(u)$ and from see (3.37), $\rho(u)$ can be approximated by $\mathbb{P}(\tau_1 \le u) = u(1 + o(1))$ as $u \to 0$.
From (3.36) we have

$$\mathbb{1}(\tau_1 \le u) \ \le \ \gamma(u) \ \le \ \mathbb{1}(\tau_1 \le u) + \gamma(u)\mathbb{1}(\tau_2 \le u) \tag{3.41}$$

and hence

$$\mathbb{P}(\tau_1 \le u) \ \le \ \mu(u) \ \le \ \mathbb{P}(\tau_1 \le u) + \mathbb{E}\gamma(u)\mathbb{1}(\tau_2 \le u).$$

From (3.37), $\mathbb{P}(0 < \tau_1 \le u) = u(1 + o(1))$ and from (3.39),

$$\mathbb{E}\gamma(u)\mathbb{1}(\tau_2 \le u) \le \mathbb{E}\gamma^2(u)\mathbb{1}(\tau_2 \le u) = O(u^2).$$

Therefore,

$$\mu(u) \; = \; u(1+o(1)) + O(u^2) \; = \; u(1+o(1)), \qquad u \to 0. \tag{3.42}$$

Similarly, for the second moment $\mu_2(u) = \mathbb{E}\gamma^2(u)$ from (3.41), (3.37), (3.39) we obtain

$$\mathbb{P}(\tau_1 \le u) \; \le \; \mu_2(u) \; \le \; \mathbb{P}(\tau_1 \le u) + 2\mathbb{E}\gamma(u)\mathbb{1}(\tau_2 \le u) + \mathbb{E}\gamma^2(u)\mathbb{1}(\tau_2 \le u),$$

implying $\mu_2(u) = u(1+o(1)) + O(u^2) = u(1+o(1))$ $(u \to 0)$ and

$$\rho(u) \; = \; \mu_2(u) - \mu^2(u) \; = \; u(1+o(1)), \qquad u \to 0. \tag{3.43}$$

Clearly, (3.42) and (3.43) imply (2.3) and (2.8). As noted in beginning of Section 2.2, (2.8) implies (2.4) for any trawl satisfying (2.9), and the existence and stationarity of the corresponding trawl process $\{X_k\}$.

(ii) Consider (2.11). Since

$$\rho(u,v) = \mathbb{E}\gamma(u)\gamma(v) - \mu(u)\mu(v) = \mathbb{E}\gamma(u)\gamma(v) - uv(1+o(1)) = \mathbb{E}\gamma(u)\gamma(v) + o(u \wedge v),$$

as $0 < u \le v \to 0$, condition (2.11) follows from

$$\mathbb{E}\gamma(u)\gamma(v) = u(1+o(1)), \qquad 0 < u \le v \to 0. \tag{3.44}$$

From (3.41) for $0 < u \le v$ we obtain

$$\begin{aligned} \mathbb{P}(\tau_1 \le u) \;\; &\le \;\; \mathbb{E}\gamma(u)\gamma(v) \\ &\le \;\; \mathbb{P}(\tau_1 \le u) + \mathbb{E}\gamma(u)\mathbb{1}(\tau_2 \le u) + \mathbb{E}\gamma(v)\mathbb{1}(\tau_1 \le u, \tau_2 \le v) \\ &\qquad + \mathbb{E}\gamma(u)\gamma(v)\mathbb{1}(\tau_2 \le u) \end{aligned}$$

where

$$\mathbb{E}\gamma(u)\gamma(v)\mathbb{1}(\tau_2 \le u) \le (\mathbb{E}\gamma^2(u)\mathbb{1}(\tau_2 \le u))^{\frac{1}{2}}(\mathbb{E}\gamma^2(v))^{\frac{1}{2}} \le Cu(\mathbb{E}\gamma^2(v))^{\frac{1}{2}}$$

and $\mathbb{E}\gamma^2(v) = \mu_2(v) = O(v)$, see (3.37), (3.39). Hence from (3.40) we have that

$$\mathbb{E}\gamma(u)\mathbb{1}(\tau_2 \le u) + \mathbb{E}\gamma(v)\mathbb{1}(\tau_1 \le u, \tau_2 \le v) + \mathbb{E}\gamma(u)\gamma(v)\mathbb{1}(\tau_2 \le u) = o(u)$$

implying (3.44) and (2.11), too. □

Theorem 2. *Assume that $a_j \ge 0$ satisfy the regular decay condition in* (2.12) *with exponent $1 < \alpha < 2$ and that the seed process in* (3.36) *satisfies conditions* (3.37)-(3.39). *Then*

$$n^{-\frac{1}{\alpha}}(S_{[nt]} - \mathbb{E}S_{[nt]}) \;\; \to_{f.d.d.} \;\; L_\alpha(t), \tag{3.45}$$

where $L_\alpha(t), t \ge 0$ is a homogeneous α-stable Lévy process with characteristic function

$$\mathbb{E}e^{\mathrm{i}zL_\alpha(t)} \;\; = \;\; \exp\left\{-t|z|^\alpha \frac{c_0\Gamma(2-\alpha)}{1-\alpha}\left(\cos(\pi\frac{\alpha}{2}) - \mathrm{i}\cdot \mathrm{sgn}(z)\sin(\pi\frac{\alpha}{2})\right)\right\}, \quad z \in \mathbb{R}. \tag{3.46}$$

Proof. Denote

$$Z = \sum_{j=0}^{\infty} \gamma(a_j), \ Z^* = \sum_{j=0}^{\infty} \mathbb{1}(\gamma(a_j) \ge 1) = \#\{j \ge 0 : a_j \ge \tau_1\}, \ Z^{**} = Z - Z^*. \tag{3.47}$$

Then $Z \ge Z^* \ge 0$ and the series for Z in (3.47) converges a.s. in view of (3.42) and has finite mean:

$$\mathbb{E}Z = \sum_{j=0}^{\infty} \mu(a_j) \le C \sum_{j=0}^{\infty} a_j < \infty.$$

We shall prove that the tail d.f. of r.v. Z decays regularly with exponent $\alpha \in (1,2)$:

$$\mathbb{P}(Z > y) \ = \ c_0\, y^{-\alpha}(1 + o(1)), \qquad \text{as } \ y \to \infty. \tag{3.48}$$

Relation (3.48) follows from (3.47) and

$$\mathbb{P}(Z^* > y) = c_0\, y^{-\alpha}(1 + o(1)), \ \text{and} \quad \mathbb{P}(Z^{**} > y) = o(y^{-\alpha}), \ \text{as } y \to \infty. \tag{3.49}$$

Consider the first relation in (3.49). Since $\mathbb{P}(Z^* > k-1) \ge \mathbb{P}(Z^* > y) \ge \mathbb{P}(Z^* > k)$ when $k-1 \le y \le k$, it suffices to show (3.49) for $y = k-1$, or the probability $\mathbb{P}(Z^* \ge k)$, $k \ge 1$. As noted in the proof of Proposition 3, for any $\epsilon > 0$ there exists $j_0 > 0$ such that $c_0(1-\epsilon)j^{-\alpha} < a_j < c_0(1+\epsilon)j^{-\alpha}$, $\forall\, j \ge j_0$. Clearly, for any $k \ge 1$ we have $\mathbb{P}(Z_- \ge k + j_0) \le \mathbb{P}(Z^* \ge k) \le \mathbb{P}(Z_+ \ge k - j_0)$, where

$$\begin{aligned} Z_+ &= \textstyle\sum_{j=j_0}^{\infty} \mathbb{1}(\tau_1 \le c_0(1+\epsilon)j^{-\alpha}) = \#\{j \ge j_0 : \tau_1 \le c_0(1+\epsilon)j^{-\alpha}\}, \\ Z_- &= \textstyle\sum_{j=j_0}^{\infty} \mathbb{1}(\tau_1 \le c_0(1-\epsilon)j^{-\alpha}) = \#\{j \ge j_0 : \tau_1 \le c_0(1-\epsilon)j^{-\alpha}\}. \end{aligned}$$

According to (3.37), as $k \to \infty$,

$$\mathbb{P}(Z_+ \ge k - j_0) = \mathbb{P}(\tau_1 < c_0(1+\epsilon)k^{-\alpha}) = \int_0^{c_0(1+\epsilon)k^{-\alpha}} \theta(y)\mathrm{d}y \sim c_0(1+\epsilon)\, k^{-\alpha}$$

and, similarly,

$$\mathbb{P}(Z_- \ge k + j_0) \ = \ \mathbb{P}(\tau_1 < c_0(1-\epsilon)(k + 2j_0 - 1)^{-\alpha}) \ \sim \ c_0(1-\epsilon)\, k^{-\alpha}.$$

Therefore, $c_0(1-\epsilon) \le \liminf k^{\alpha}\mathbb{P}(Z^* \ge k) \le \limsup k^{\alpha}\mathbb{P}(Z^* \ge k) \le c_0(1+\epsilon)$, where $\epsilon > 0$ is arbitrary small, proving the first fact in (3.49). To prove the second fact in (3.49), note $Z^{**} \le \sum_{j=0}^{\infty} \gamma(a_j)\mathbb{1}(a_j \ge \tau_2)$ and then by (3.39) and Minkowski's inequality we obtain

$$\mathbb{E}^{\frac{1}{2}}(Z^{**})^2 \quad \le \quad \sum_{j=0}^{\infty} \left(\mathbb{E}\gamma^2(a_j)\mathbb{1}(a_j \ge \tau_2)\right)^{\frac{1}{2}} \ \le \ C \sum_{j=0}^{\infty} |a_j| \ < \ \infty$$

proving (3.49) and hence (3.48) as well. In turn, (3.48) implies that the distribution of r.v. Z belongs to the domain of attraction of asymmetric α-stable law, viz.,

$$n^{-\frac{1}{\alpha}} \sum_{k=1}^{[nt]} (Z_k - \mathbb{E}Z_k) \ \to_{f.d.d.} \ L_\alpha(t), \tag{3.50}$$

where $Z_k = \sum_{j=0}^{\infty} \gamma_k(a_j)$, $k \in \mathbb{Z}$ are i.i.d. copies of r.v. Z in (3.47) and L_α is the α-stable Lévy process in (3.46). See e.g. ([14], Theorem 2.6.7).

We now use the decomposition Lemma 1 and its notations. The convergence in (3.45) will follow from (3.50) if we show that the partial sums process in (3.45) can be approximated by the partial sums process in (3.50), in the sense that

$$\mathbb{E}|S_n - \widetilde{S}_n| \ = \ o(n^{\frac{1}{\alpha}}), \qquad \text{where} \quad \widetilde{S}_n = \sum_{k=1}^{n} Z_k. \tag{3.51}$$

Indeed,

$$n^{-\frac{1}{\alpha}}(S_{[nt]} - \mathbb{E}S_{[nt]}) = n^{-\frac{1}{\alpha}}(\widetilde{S}_{[nt]} - \mathbb{E}\widetilde{S}_{[nt]}) + R_{[nt]},$$

where $R_n = n^{-\frac{1}{\alpha}}(S_n - \widetilde{S}_n) + n^{-\frac{1}{\alpha}}\mathbb{E}(\widetilde{S}_n - S_n) = o_{\mathbb{P}}(1)$ according to (3.51). We have $\widetilde{S}_n - S_n = R'_n - R''_n$, where

$$R'_n = \sum_{1\le s\le n} \sum_{j>n-s} \gamma_s(a_j), \qquad R''_n = \sum_{s\le 0} \sum_{1\le k\le n} \gamma_s(a_{k-s}),$$

then $R'_n \ge 0, R''_n \ge 0$.

Using (3.42) and (2.12) we obtain

$$\begin{aligned}
\mathbb{E}R'_n \ &= \ \sum_{1\le s\le n} \sum_{j>n-s} \mathbb{E}\gamma_s(a_j) \ = \ \sum_{1\le s\le n} \sum_{j>n-s} \mu(a_j) \\
&\le \ C \sum_{1\le s\le n} \sum_{j>n-s} j^{-\alpha} \ = \ O(n^{2-\alpha}), \\
\mathbb{E}R''_n \ &= \ \sum_{s\le 0} \sum_{1\le k\le n} \mathbb{E}\gamma_s(a_{k-s}) \ = \ \sum_{s\le 0} \sum_{1\le k\le n} \mu(a_{k-s}) \\
&= \ \sum_{s\ge 0} \sum_{1\le k\le n} \frac{1}{(k+s)^\alpha} \ = \ O(n^{2-\alpha}),
\end{aligned}$$

implying (3.51) since $2-\alpha < 1/\alpha$ for $1 < \alpha < 2$. Theorem 2 is proved. □

Example 8 (Jump processes satisfying the assumptions of Theorem 2)**.** Note that for a jump process in (3.36) we have $(\gamma(u) = k) = (\tau_k \le u < \tau_{k+1})$ since the sets on the r.h.s. of (3.36) are disjoint. Conditions in (3.37)-(3.40) on the seed process $\{\gamma(u), u \ge 0\}$ in Theorem 2 are rather weak and essentially involve the distribution of the first jump-point τ_1 provided the second jump τ_2 cannot occur very fast after τ_1. Particularly,

- The Bernoulli seed process of Example 4 can be written in the form (3.36), where $\tau_1 \sim \mathcal{U}[0,1]$ is uniformly distributed and $\tau_k = \infty$, for $k \ge 2$. In this case, (3.37) holds with $\theta(u) = \mathbb{1}(u \le 1)$ while (3.38)-(3.40) are trivially satisfied by $\tau_2 = \infty$ a.s., which means that the sum (3.36) contains two terms only.

- The binomial seed process $b(u;n) = \sum_{j=1}^{n} \mathbb{1}(U_j \le u)$ (see Example 4) can be represented as (3.36) with $\tau_1 = \min\{U_j : 1 \le j \le n\}$ and τ_j is the j-th order statistic of $U_1, \dots, U_n$, if $2 \le j \le n$. Finally $\tau_{n+1} = \infty$. Thus, the probability density of the joint distribution of $(\tau_j, 1 \le j \le n)$ equals $\theta(u_1, \dots, u_n) = n! \mathrm{d}u_1 \cdots \mathrm{d}u_n \mathbb{1}(0 < u_1 < \cdots < u_n < 1)$. Particularly, $\theta(u) = \mathbb{P}(\tau_1 \in \mathrm{d}u)/\mathrm{d}u = n(1-u)^{n-1}$ satisfies $\lim_{u\to 0} \theta(u) = n$, or condition (3.37) with 1 replaced by n, while $\theta(u_1, u_2) = (1/2)n(n-1)(1-u_2)^{n-2}\mathbb{1}(0 < u_1 < u_2 < 1)$. Since $\gamma(u) = b(u;n) \le n$, condition (3.38) is trivially satisfied and (3.39) follows by $\mathbb{E}\gamma(u)\mathbb{1}(\theta_2 \le u) \le n\mathbb{P}(\theta_2 \le u) \le (n/2)n(n-1)\int_{0<u_1<u_2\le u} \mathrm{d}u_1 \mathrm{d}u_2 = (n/4)n(n-1)u^2 = O(u^2)$. Relation (3.40) follows similarly from

$$\begin{aligned}
\mathbb{E}\gamma(v)\mathbb{1}(\tau_1 \le u, \tau_2 \le v) &\le n\mathbb{P}(\tau_1 \le u, \tau_2 \le v) \\
&\le (n/2)n(n-1)\int_{0<u_1\le u, u_1<u_2\le v} \mathrm{d}u_1 \mathrm{d}u_2 \\
&\le (n/2)n(n-1)uv = o(u), \quad 0 < u < v \to 0.
\end{aligned}$$

- The Poisson process $\gamma(u) = P(u)$ of Example 3 can be written as (3.36) with i.i.d. $\tau_1, \tau_k - \tau_{k-1}, k \ge 2$ distributed according to the exponential law with density $\theta(u) = \mathrm{e}^{-u}, u > 0$. In this case, (3.37) is satisfied and (3.39) holds since

$$\begin{aligned}
\mathbb{E}P(u)^2\mathbb{1}(\tau_2 \le u) &= \mathbb{E}P(u)^2 - \mathbb{P}(P(u) = 1) \\
&= u + u^2 - u\mathrm{e}^{-u} = O(u^2).
\end{aligned} \tag{3.52}$$

  Condition (3.40) can be also directly verified for $\gamma(u) = P(u)$ using properties of Poisson process. The Poisson process is a particular case of generalized renewal process defined below.

- A generalized renewal process is a jump process γ in (3.36) such that r.v.s $U_k = \tau_k - \tau_{k-1}$, $k \ge 1$ (intervals between successive jumps) are independent. Sufficient assumptions on the distribution of $U_k, k \ge 1$ guaranteeing (3.37)-(3.40) for such process are given in Proposition 5 below.

- The mixed Poisson process $\gamma(u) = P(\zeta u)$ of Example 3 also satisfies (3.37)-(3.40) under mild conditions on the mixing r.v. ζ. See Proposition 6.

Proposition 5. *Let* $\gamma = \{\gamma(u), u \ge 0\}$ *in* (3.36) *be a generalized renewal process such that the lengths* $U_k = \tau_k - \tau_{k-1}, k \ge 1$ *between successive jumps of* (3.36) *have uniformly bounded probability densities, viz.,*

$$\mathbb{P}(U_k \le u) = \int_0^u \theta_k(y)\mathrm{d}y, \ k \ge 1, \quad \text{with} \quad \sup_{k\ge 1, u \ge 0} \theta_k(u) \le K < \infty. \tag{3.53}$$

Moreover, assume $\lim_{u\to 0} \theta_1(u) = 1$. *Then* γ *satisfies conditions* (3.37)-(3.40).

Proof. Condition (3.37) is obviously satisfied. Consider (3.38). We use the representation

$$\gamma(u) = \sum_{k=1}^{\infty} \mathbb{1}(\tau_k \le u) = \sum_{k=1}^{\infty} \mathbb{1}(U_1 + \cdots + U_k \le u) \qquad (3.54)$$

see ([4], chapter 23, p. 307). Then by Minkowski's inequality and (3.53), (3.54) for any $u \ge 0$ we obtain

$$\begin{aligned}
\mathbb{E}\gamma(u)^{2+\delta} &\le \Big(\sum_{k=1}^{\infty} \mathbb{P}(\tau_k \le u)^{1/(2+\delta)}\Big)^{2+\delta} \\
&= \Big(\sum_{k=1}^{\infty} \Big\{ \int_{\mathbb{R}_+^k} \mathbb{1}(z_1 + \cdots + z_k \le u) \prod_{i=1}^{k} \theta_i(z_i) \mathrm{d}z_i \Big\}^{1/(2+\delta)}\Big)^{2+\delta} \\
&\le \Big(\sum_{k=1}^{\infty} \Big\{ K^k \int_{\mathbb{R}_+^k} \mathbb{1}(z_1 + \cdots + z_k \le u) \prod_{i=1}^{k} \mathrm{d}z_i \Big\}^{1/(2+\delta)}\Big)^{2+\delta} \\
&= \Big(\sum_{k=1}^{\infty} \Big\{ \frac{K^k u^k}{k!} \Big\}^{1/(2+\delta)}\Big)^{2+\delta} < \infty
\end{aligned}$$

proving (3.38). Next, using $\mathbb{1}(\tau_j \le u)\mathbb{1}(\tau_k \le u) = \mathbb{1}(\tau_k \le u)$, $j \le k$

$$\begin{aligned}
\mathbb{E}\gamma(u)^2 \mathbb{1}(\tau_2 \le u) &= \mathbb{E}\Big(\sum_{k=1}^{\infty} \mathbb{1}(\tau_k \le u)\Big)^2 \mathbb{1}(\tau_2 \le u) \\
&= \mathbb{E}\Big(\sum_{k=1}^{\infty} k \cdot \mathbb{1}(\tau_k \le u)\Big) \mathbb{1}(\tau_2 \le u) \\
&\le \sum_{k=2}^{\infty} (k+1) \int_{\mathbb{R}_+^k} \mathbb{1}(z_1 + \cdots + z_k \le u) \prod_{i=1}^{k} \theta_i(z_i) \mathrm{d}z_i \\
&\le \sum_{k=2}^{\infty} K^k (k+1) \int_{\mathbb{R}_+^k} \mathbb{1}(z_1 + \cdots + z_k \le u) \prod_{i=1}^{k} \mathrm{d}z_i \\
&= \sum_{k=2}^{\infty} \frac{K^k (k+1) u^k}{k!} = O(u^2)
\end{aligned}$$

hence (3.39) holds. Finally, $\mathbb{E}\gamma(v)\mathbb{1}(\tau_1 \le u, \tau_2 \le v) = \mathbb{E}\Big(\sum_{k=1}^{\infty} \mathbb{1}(\tau_k \le v)\Big)\mathbb{1}(\tau_1 \le u, \tau_2 \le v) = \mathbb{P}(\tau_1 \le u, \tau_2 \le v) + \sum_{k=2}^{\infty} \mathbb{P}(\tau_k \le v, \tau_1 \le u, \tau_2 \le v)$, where

$\mathbb{P}(\tau_1 \le u, \tau_2 \le v) = \mathbb{P}(U_1 \le u, U_1 + U_2 \le v) \le \mathbb{P}(U_1 \le u, U_2 \le v) \le K^2 uv$ and

$$\begin{aligned}\sum_{k=2}^{\infty} \mathbb{P}(\tau_k \le v, \tau_1 \le u, \tau_2 \le v) &\le \sum_{k=2}^{\infty} \mathbb{P}(\tau_1 \le u, \tau_k - \tau_1 \le v) \\ &= \mathbb{P}(U_1 \le u) \sum_{k=2}^{\infty} \mathbb{P}(U_2 + \cdots + U_k \le v) \\ &\le Ku \sum_{k=1}^{\infty} K^k \int_{\mathbb{R}_+^k} \mathbb{1}(z_1 + \cdots + z_k \le v) \prod_{i=1}^{k} \mathrm{d}z_i \\ &\le Ku \sum_{k=1}^{\infty} \frac{K^k v^k}{k!} \le Cuv,\end{aligned}$$

implying $\mathbb{E}\gamma(v)\mathbb{1}(\tau_1 \le u, \tau_2 \le v) \le Cuv = o(u)$ as $0 \le u \le v \to 0$, or (3.40). □

Proposition 6. *Let $\gamma = \{P(\zeta u), u \ge 0\}$ be a mixed Poisson process of Example 3, where $\zeta > 0$ is independent of Poisson process P and satisfies $\mathbb{E}\zeta = 1$, and $\mathbb{E}\zeta^{2+\delta} < \infty$ for some $\delta > 0$. Then γ satisfies conditions* (3.37)-(3.40).

Proof. Condition (3.37) follows from $\mathbb{P}(\tau_1 \le u) = \mathbb{E} \int_0^{\zeta u} \mathrm{e}^{-y} \mathrm{d}y \sim \mathbb{E}\zeta = 1 \, (u \to 0)$. To show (3.38) we need the bound $EP(u)^{2+\delta} \le 5(u \vee u^{2+\delta}) \, (u > 0)$. To get it recall that $\mathbb{E}(P(u) - u)^3 = u$ thus

$$\mathbb{E}P(u)^3 = u + 3u^2 + u^3 \le 4(u + u^3), \tag{3.55}$$

and Jensen inequality yields the result $\mathbb{E}P(u)^{2+\delta} \le 5^{\frac{2+\delta}{3}} u^{2+\delta}$ for $u \ge 1$, and if $u \le 1$ simply quote that, since the Poisson process admits integer values, $\mathbb{E}P(u)^{2+\delta} \le \mathbb{E}P(u)^3 \le 5u$. Whence, $\mathbb{E}\zeta(u)^{2+\delta} \le C\mathbb{E}\zeta^{2+\delta} u^{2+\delta} < \infty$ proving (3.38). Similarly using (3.52) we get

$$\mathbb{E}\zeta(u)^2 \mathbb{1}(\tau_2 \le u) = u^2 \mathbb{E}\zeta^2 + u\mathbb{E}[\zeta(1 - \mathrm{e}^{-\zeta u})] \le 2u^2 \mathbb{E}\zeta^2 = O(u^2) \qquad (u \to 0),$$

proving (3.39). To show (3.40) we use a suitable bound for Poisson process:

$$\mathbb{E}P(v)\mathbb{1}(\tau_1^* \le u, \tau_2^* \le v) \le C\big[(u(1 - \mathrm{e}^{-u}))^{2/3}(v^{1/3} + v) + uv\big], \qquad 0 < u < v < \infty \tag{3.56}$$

which is valid for all $0 < u < v < \infty$ and where $\tau_j^*, j \ge 1$ are jump times of the Poisson process $P(u) = \sum_{j=1}^{\infty} \mathbb{1}(\tau_j^* \le u)$. Let $q(u, v)$ denote the l.h.s. of (3.56), then $q(u, v) = q_1(u, v) + q_2(u, v)$, with $q_1(u, v) = \mathbb{E}P(v)\mathbb{1}(\tau_2^* \le u) \le \mathbb{P}^{2/3}(\tau_2^* \le u)\mathbb{E}^{1/3} P(v)^3$, where $\mathbb{P}(\tau_2^* \le u) = \mathbb{P}(P(u) \ge 2) = 1 - \mathrm{e}^{-u}(1+u) \le u(1 - \mathrm{e}^{-u})$ and, from (3.55), $\mathbb{E}P(v)^3 \le 4[v + v^3]$. Therefore, $q_1(u, v)$ does not exceed the r.h.s. of (3.56). Next, since $\tau_2^* > u$ implies $P(u) = 1$ we obtain $q_2(u, v) = \mathbb{E}P(v)\mathbb{1}(\tau_1^* \le u, u < \tau_2^* \le v) = \mathbb{P}(\tau_1^* \le u, u < \tau_2^* \le v) + \mathbb{E}(P(v) - P(u))\mathbb{1}(P(u) = 1, P(v) \ge 1)$, where $\mathbb{P}(\tau_1^* \le u, u < \tau_2^* \le v) = \mathbb{P}(P(u) = 1, P(v) - P(u) \ge 1) = u\mathrm{e}^{-u}(1 - \mathrm{e}^{-(v-u)}) \le u(v - u) \le uv$ and similarly, $\mathbb{E}(P(v) - P(u))\mathbb{1}(P(u) = 1, P(v) \ge$

1) = $\mathbb{P}(P(u) = 1)\mathbb{E}P(v-u) = u\mathrm{e}^{-u}(v-u) \le uv$, thus proving (3.56). With (3.56) in mind, we obtain

$$\begin{aligned}
&\mathbb{E}\gamma(v)\mathbb{1}(\tau_1 \le u, \tau_2 \le v) \ = \mathbb{E}P(\zeta v)\mathbb{1}(\tau_1^* \le \zeta u, \tau_2^* \le \zeta v) \\
&\le \ C\mathbb{E}\big[(\zeta u(1-\mathrm{e}^{-\zeta u}))^{2/3}((\zeta v)^{1/3} + (\zeta v)) + \zeta^2 uv\big] \\
&\le \ C\Big\{\mathbb{E}^{1/2}[(\zeta u)^{4/3}(1-\mathrm{e}^{-\zeta u})^{4/3}]\mathbb{E}^{1/2}[(\zeta v)^{2/3} + (\zeta v)^2] + uv\mathbb{E}\zeta^2\Big\} \\
&\le \ C\big\{u^{2/3}\mathbb{E}^{1/2}[\zeta^{4/3}(1-\mathrm{e}^{-\zeta u})^{4/3}](v^{1/3} + v) + uv\big\}.
\end{aligned}$$

Hence, (3.40) follows if we show that $\mathbb{E}\zeta^2 < \infty$ implies

$$\mathbb{E}[\zeta^{4/3}(1-\mathrm{e}^{-\zeta u})^{4/3}] \le Cu^{2/3}. \tag{3.57}$$

To prove this recall that for $v \ge 0$, $1 - e^{-v} \le v \wedge 1$, thus $(1-\mathrm{e}^{-\zeta u})^{4/3} \le (\zeta u)^{2/3}1^{2/3}$ and the inequality is proved with $C = \mathbb{E}\zeta^2$. Proposition 6 is proved. □

3.3. Functional convergence in $D[0,1]$

Let us note that functional convergence in (3.45) is a delicate problem and may not hold in the usual J_1-topology. See [17], [26], [29], [24] on weak convergence in $D[0,1]$ with stable limits. In this subsection at the cost of additional structure (association property) we prove the weak convergence of the partial sums process in (3.45) in Skorohod's M_1-topology (see [31] or [34]).

Recall that r.v.s $V_1, V_2, \dots, V_m$ are said *associated* if

$$\mathrm{Cov}\big(f(V_1, V_2, \dots, V_m), g(V_1, V_2, \dots, V_m)\big) \ge 0$$

for all nondecreasing functions f and g for which the covariance exists. An infinite family of r.v.s is associated if its every finite subfamily is associated. Association of r.v.s was introduced in Esary et al. [10] and we refer to this paper for basic properties of this notion. We only recall the useful statements that independent r.v.s are associated and also the heredity of this notion through non-decreasing functions.

A simple application of these properties entails the following lemma.

Lemma 2. *If the seed process $\{\gamma(u), u \in \mathbb{R}\}$ is associated, then the trawl process $\{X_k, k \in \mathbb{Z}\}$ in (1.1) is associated.*

Theorem 3. *Suppose that all the assumptions of Theorem 2 hold. In addition, if the jump times $\tau_1, \tau_2, \dots$ are associated (e.g., if all τ_ks are sums of independent positive r.v.s) then $\{X_k\}$ is associated and the finite-dimensional convergence (3.45) can be strengthened to*

$$n^{-\frac{1}{\alpha}}(S_{[nt]} - \mathbb{E}S_{[nt]}) \ \to_{\mathcal{D}(M_1)} \ L_\alpha(t), \tag{3.58}$$

Proof. Since (3.58) follows from the association of $\{X_k\}$ and a general result in Louichi and Rio ([24], Theorem 1), by Lemma 2 it suffices to verify that the seed

$\gamma = \{\gamma(u),\, u \ge 0\}$ is associated, i.e. r.v.s $\gamma(u_1), \gamma(u_2), \ldots, \gamma(u_m)$ are associated for any $m \ge 1$ and any $0 < u_1 < \cdots < u_m < \infty$. Using the representation of $\gamma(u)$ in (3.54) and the arguments already presented above it is enough to prove the association of r.v.s

$$\begin{array}{cccc} \mathbb{1}(\tau_1 \le u_1), & \mathbb{1}(\tau_2 \le u_1), & \cdots, & \mathbb{1}(\tau_k \le u_1), \\ \mathbb{1}(\tau_1 \le u_2), & \mathbb{1}(\tau_2 \le u_2), & \cdots, & \mathbb{1}(\tau_k \le u_2), \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{1}(\tau_1 \le u_m), & \mathbb{1}(\tau_2 \le u_m), & \cdots, & \mathbb{1}(\tau_k \le u_m), \end{array} \tag{3.59}$$

for any $k \ge 1$. Let us notice that $\mathbb{1}(\tau_j \le u_i) = 1 - \mathbb{1}(\tau_j > u_i)$ and that functions $x \mapsto f_i(x) = \mathbb{1}(x > u_i), 1 \le i \le m$ are nondecreasing. Since association is preserved by nondecreasing transformations, the association of $\{\tau_1, \ldots, \tau_m\}$ implies the same property of $\{f_i(\tau_j), 1 \le i \le m, 1 \le j \le k\}$. By property BP_1 of binary r.v.s in [10], $\{1 - f_i(\tau_j), 1 \le i \le m, 1 \le j \le k\}$, or array (3.59) is associated as well, ending the proof of Theorem 3. □

Remark 4. Association of γ or jump times $\{\tau_j\}$ can be easily verified in most examples considered in this paper. In particular,

- Poisson and generalized renewal processes (see Example 8) are associated since $\tau_k = U_1 + \cdots + U_k$ is a sum of independent r.v.s.
- For a mixed Poisson process with $\gamma > 0$ we have $\tau_k = (E_1 + \cdots + E_k)/\zeta$, where $E_j, j \ge 1$ are independent exponentially distributed r.v.s. Thus, $\{\tau_j,\, j \ge 1\}$ is associated, the latter being nondecreasing transformations of independent r.v.s $\{1/\zeta, E_j,\, j \ge 0\}$. The above observation extends to a general mixed Poisson process with $\gamma \ge 0$.
- Bernoulli seed in Example 4 is associated as it follows from the proof of Theorem 3. The binomial seed of the same example is also associated, being the sum of n independent associated processes, see ([10], Property P_2). Alternatively, association of the binomial seed can be established by showing that the order statistics are nondecreasing functions of (independent uniformly distributed) sample variables.

Remark 5. Theorem 2 and the subsequent discussion refers to trawl processes with values in $\mathbb{N}$ corresponding to seed process with positive jumps, in which case the limit α-stable process is completely asymmetric. Clearly, this result can be extended to some trawl processes with values in $\mathbb{Z}$ and a symmetric limit distribution. Particularly, if $\gamma = \gamma^+ - \gamma^-$ is the difference of two independent copies of jump processes of the form (3.36), the corresponding trawl process in (1.1) also writes as the difference $X_k = X_k^+ - X_k^-$ of two independent trawl processes with values in $\mathbb{N}$ and the limit distribution of $S_n = \sum_{k=1}^n X_k$ can be symmetric α-stable. More generally, consider two families $\{\gamma_j^+\}, \{a_j^+\}$ and $\{\gamma_j^-\}, \{a_j^-\}$ and the corresponding trawl processes $\{X_k^+\}$ and $\{X_k^-\}$ and partial sum processes $S^+_{[nt]}$ and $S^-_{[nt]}$. If the sequences $\{\gamma^+_j\}$ and $\{\gamma^-_j\}$ are mutually independent and satisfy the conditions of Theorem 2, then the convergence

$$n^{-\frac{1}{\alpha}}(S^+_{[nt]}-\mathbb{E}S^+_{[nt]}) \to_{f.d.d.} L^+_\alpha(t), \quad n^{-\frac{1}{\alpha}}(S^-_{[nt]}-\mathbb{E}S^-_{[nt]}) \to_{f.d.d.} L^-_\alpha(t), \tag{3.60}$$

implies also

$$n^{-\frac{1}{\alpha}}\Big(\big(S^+_{[nt]} - S^-_{[nt]}\big) - \mathbb{E}\big(S^+_{[nt]} - S^-_{[nt]}\big)\Big) \to_{f.d.d.} L_\alpha(t), \tag{3.61}$$

where the α-stable Lévy process L_α has the same distribution as the difference $L^+_\alpha - L^-_\alpha$ of independent α-stable Lévy processes L^+_α and L^-_α.

Remark 6. Suppose we are able to strengthen the convergence in (3.60) to the functional convergence in the M_1 topology (e.g. by applying Theorem 3). Then we cannot *automatically* replace (3.61) with the functional convergence in the M_1 topology due to the lack of continuity of addition in M_1 (see e.g. [34]). However, the desired convergence can be achieved if we go deeper into the properties of M_1. Lemma 3 seems to be known, but we could not find any reference matching our framework. For the sake of completeness we decided to include the proof.

Lemma 3. *Suppose that for each n, càdlàg processes Z'_n and Z''_n are* independent *and, as $n \to \infty$,*

$$Z'_n(t) \to_{\mathcal{D}(M_1)} L'(t), \quad Z''_n(t) \to_{\mathcal{D}(M_1)} L''(t), \tag{3.62}$$

where $L'(t)$ and $L''(t)$ are homogeneous Lévy processes *without Gaussian component. Then*

$$Z'_n(t) + Z''_n(t) \to_{\mathcal{D}(M_1)} L'_0(t) + L''_0(t), \tag{3.63}$$

where $L'_0(t)$ and $L''_0(t)$ are independent copies of $L'(t)$ and $L''(t)$, respectively.

Proof. Passing to an a.s Skorokhod representation (see e.g. [9]) on the product space

$$\big(D[0,1], M_1\big) \times \big(D[0,1], M_1\big),$$

we may and do assume that for each $\omega \in \Omega$

$$Z'_n(\cdot,\omega) \to_{M_1} L'_0(\cdot,\omega), \quad Z''_n(\cdot,\omega) \to_{M_1} L''_0(\cdot,\omega),$$

with Z'_n and Z''_n independent for $n = 1, 2, \ldots$, and L'_0 and L''_0 being independent copies of L' and L''. Here $\to_{M_1}$ denotes the convergence in $D[0,1]$ equipped with the M_1 topology. We claim that it is enough to prove that *almost surely*

$$\mathrm{Disc}(L'_0) \bigcap \mathrm{Disc}(L''_0) = \emptyset, \tag{3.64}$$

where for a càdlàg function x

$$\mathrm{Disc}(x) = \big\{t \in [0,1]\,;\ \Delta x_t = x_t - x_{t-} \neq 0\big\}.$$

Indeed, we would have then by corollary 12.7.1 in [34] that *almost surely*

$$Z_n'(\cdot,\omega) + Z_n''(\cdot,\omega) \to_{M_1} L_0'(\cdot,\omega) + L_0''(\cdot,\omega),$$

what implies (3.63) .

Relation (3.64) follows from ([6], Proposition 5.3) and the fact that the Lévy measure of (L_0', L_0'') is concentrated on the coordinate axes, since in this case for almost all ω, the jumps satisfy

$$\Delta L_0'(\cdot,\omega)_t \cdot \Delta L_0''(\cdot,\omega)_t = 0, \quad t \in [0,1],$$

as desired. □

Corollary 1. *Let $\{X_k^+\}$ and $\{X_k^-\}$ be trawl processes built according to recipe (1.1), using systems $\{\gamma_j^+\}, \{a_j^+\}$ and $\{\gamma_j^-\}, \{a_j^-\}$, respectively, and let $S_{[nt]}^+$ and $S_{[nt]}^-$ be the corresponding partial sum processes.*

Suppose that $\{\gamma_j^+\}$ and $\{\gamma_j^-\}$ are mutually independent and both satisfy the assumptions of Theorem 3. Let L_α^+ and L_α^- be the limiting α-stable Lévy processes for $n^{-\frac{1}{\alpha}}(S_{[nt]}^+ - \mathbb{E}S_{[nt]}^+)$ and $n^{-\frac{1}{\alpha}}(S_{[nt]}^- - \mathbb{E}S_{[nt]}^-)$, respectively. Then we have

$$n^{-\frac{1}{\alpha}}\Big(\big(S_{[nt]}^+ - S_{[nt]}^-\big) - \mathbb{E}\big(S_{[nt]}^+ - S_{[nt]}^-\big)\Big) \to_{\mathcal{D}(M_1)} L_\alpha(t), \tag{3.65}$$

where the α-stable Lévy process L_α has the same distribution as the difference $L_\alpha' - L_\alpha''$ of independent copies of L_α^+ and L_α^-.

Remark 7. The example of an ordinary moving average with summable coefficients shows that (3.60) may imply (3.61) *without the assumption of independence* of $S_{[nt]}^+$ and $S_{[nt]}^-$ (see e.g. [1], corollary 2.2). In the functional limit result given below we follow this general approach and obtain the functional convergence in the non-Skorohodian S topology (see [15]). We shall denote by $\to_{\mathcal{D}(S)}$ the convergence in distribution on the Skorohod space $D[0,1]$ equipped with the S topology.

Corollary 2. *As in Corollary 1 we consider systems $\{\gamma_j^+\}, \{a_j^+\}, \{X_k^+\}, \{S_{[nt]}^+\}$ and $\{\gamma_j^-\}, \{a_j^-\}, \{X_k^-\}, \{S_{[nt]}^-\}$, each satisfying the conditions of Theorem 3, so that*

$$\begin{aligned} n^{-\frac{1}{\alpha}}(S_{[nt]}^+ - \mathbb{E}S_{[nt]}^+) &\to_{\mathcal{D}(M_1)} L_\alpha^+(t), \\ n^{-\frac{1}{\alpha}}(S_{[nt]}^- - \mathbb{E}S_{[nt]}^-) &\to_{\mathcal{D}(M_1)} L_\alpha^-(t). \end{aligned} \tag{3.66}$$

Allowing dependence between $\{\gamma_j^+\}$ and $\{\gamma_j^-\}$ we assume that for some càdlàg stochastic process K we have

$$n^{-\frac{1}{\alpha}}\Big(\big(S_{[nt]}^+ - S_{[nt]}^-\big) - \mathbb{E}\big(S_{[nt]}^+ - S_{[nt]}^-\big)\Big) \to_{f.d.d.} K(t). \tag{3.67}$$

Then

$$n^{-\frac{1}{\alpha}}\Big(\big(S_{[nt]}^+ - S_{[nt]}^-\big) - \mathbb{E}\big(S_{[nt]}^+ - S_{[nt]}^-\big)\Big) \to_{\mathcal{D}(S)} K(t).$$

Proof. Let us notice that the S topology is sequential, but non-metric, and therefore standard (for the metric case) steps require some more subtle arguments. This is the reason why we provide exact reference to each step in the proof.

First, the topology M_1 is stronger than S, hence (3.66) implies the uniform S-tightness of the corresponding processes (for details see [1], theorem 3.13). By the sequential continuity of addition in the S topology, the differences $n^{-\frac{1}{\alpha}}\big((S^+_{[nt]} - S^-_{[nt]}) - (\mathbb{E}S^+_{[nt]} - \mathbb{E}S^-_{[nt]})\big)$ are also uniformly S-tight (see [1], proposition 3.16).

Thus we have uniform S-tightness and finite dimensional convergence (3.67), which imply the functional convergence in S (see [1], proposition 3.3). □

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